Abstract Algebra IV
Final Exam

**Note:** You may use any result proved in class, *unless* the question is asking you to (re)prove a result that we have already proved in class! On the other hand, if you need to use the result of a *homework problem* to answer a question below, then you must put down the solution to that homework problem also (i.e., you cannot simply quote the result of a homework problem).

**Note:** Each problem is worth the same amount.

**Remark:** You have seen almost all these problems before in your homework.

(1) Let $f(x)$ be an irreducible polynomial of degree $n$ with coefficients in a field $F$, and let $K$ be an extension field of $F$. Assume that $[K : F]$ is finite, and relatively prime to $n$. Show that $f(x)$ remains irreducible in $K[x]$. Use this to show that $x^5 - 9x^3 + 15x + 6$ is irreducible over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(2) Let $K$ be a (finite) Galois extension of $F$, and let $a \in K$. Let $n = [K : F]$, $r = [F(a) : F]$, $G = \mathrm{Gal}(K/F)$ and $H = \mathrm{Gal}(K/F(a))$. Let $\tau_1, \ldots, \tau_r$ be left coset representatives of $H$ in $G$. Show that the minimum polynomial $m_{a,F}$ of $a$ over $F$ is $\prod_{i=1}^{r}(x - \tau_i(a))$. Conclude that $\prod_{g\in G}(x - g(a)) = (m_{a,F})^{n/r}$.

(3) Let $K = \mathbb{Q}(\omega_n)$, where $\omega_n$ is the primitive $n$-th root of unity $e^{2\pi\imath/n}$ $(n \geq 3)$.
   (a) Show that the fixed field of $K$ under complex conjugation is $\mathbb{Q}(\tau_n)$, where $\tau_n = \omega_n + \omega_n^{-1}$.
   (b) Now take $n$ to be of the form $2^{m+2}$, $m \geq 1$. Show that $\mathbb{Q}(\tau_n)$ is

$$\mathbb{Q}\left(\sqrt{2+\sqrt{2+\sqrt{\cdots+\sqrt{2}}}}\right)$$

   where the square root is taken $m$ times.

(4) Show that every element in a finite field is a sum of two squares.

(5) Let $K$ be an algebraically closed field and $F$ a subfield of $K$. If $\phi\colon K \mapsto K$ is an $F$-homomorphism, and if $\mathrm{trdeg}(K/F) < \infty$, show that $\phi$ is surjective (so $\phi$ is an $F$-automorphism of $K$).

(6) Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Given $a \in \overline{\mathbb{Q}}$, let $L$ be a subfield of $\overline{\mathbb{Q}}$ that is maximal with respect to *not containing* $a$. (Thus, if $E$ is a subfield of $\overline{\mathbb{Q}}$ that strictly contains $L$, then $E$ contains $a$.) Show that
   (a) $L(a)$ is a finite extension of $L$.
   (b) $L(a)/L$ is normal. (Hint: If $a'$ is another root of $m_{a,L}$, what can you say about $L(a')$?)
   (c) Show that $L(a)/L$ is cyclic, of prime degree. (Hint: Apply the Galois correspondence to proper subgroups of $\mathrm{Gal}(L(a)/L)$.)